number of triads of a graph. $\mathbb{E}(\frac{1}{d_i d_j d_k})$ is the expected value of $\frac{1}{d_i d_j d_k}$ over all triads. Hence, $m_{3,H} = 2\,\mathbb{E}_H(\Delta_i)\,\mathbb{E}_H(\frac{1}{d_i d_j d_k})$. For any triad (i, j, k) in G, it has $n-3$ copies in which no member of i, j, k is removed, so $\mathbb{E}_H(\Delta_i) = \frac{3\Delta_H}{|V_H|} = \frac{3(n-3)\Delta_G}{n(n-1)} = \frac{n-3}{n-1}\,\mathbb{E}_G(\Delta_i)$.

To get $\mathbb{E}_H(\frac{1}{d_i d_j d_k})$, we consider four types of triads based on their node degrees.

► **Type I**: $d_i = 2, d_j = 2, d_k = 2$. For such a triad, all of its $n-3$ copies have $\frac{1}{d_i d_j d_k} = \frac{1}{8}$ as neither of i and j can lose other neighbors, so there will be no increment;

► **Type II**: $d_i > 2, d_j = 2, d_k = 2$. For a triad of this type, if the removed node is a neighbor of i, then the triad contributes an increment $\frac{1}{(d_i-1)d_j d_k} - \frac{1}{d_i d_j d_k} = \frac{1}{d_i d_j d_k (d_i-1)}$. Among the $n-3$ copies, there are $d_i - 2$ such cases, so the overall contribution is $(d_i - 2)\cdot\frac{1}{d_i d_j d_k (d_i-1)} = \frac{1}{d_i d_j d_k}\cdot\frac{d_i-2}{d_i-1} < \frac{1}{d_i d_j d_k}$.

► **Type III**: $d_i > 2, d_j > 2, d_k = 2$. If the removed node is a neighbor of i but not connected to j (or a neighbor of j but not connected to i) and not including k, the triad contributes an increment $\frac{1}{(d_i-1)d_j d_k} - \frac{1}{d_i d_j d_k} = \frac{1}{d_i d_j d_k (d_i-1)}$ (or $\frac{1}{d_i d_j d_k (d_j-1)}$). If the removed node is a common neighbor of i and j, the increment is $\frac{1}{(d_i-1)(d_j-1)d_k} - \frac{1}{d_i d_j d_k} = \frac{1}{d_i d_j d_k (d_i-1)} + \frac{1}{d_i d_j d_k (d_j-1)} + \frac{1}{d_i d_j d_k (d_i-1)(d_j-1)}$. Assume i and j have c_{ij} common neighbors not including k, then the overall increment of a triad of Type III is $(d_i - 2 - c_{ij})\cdot\frac{1}{d_i d_j d_k (d_i-1)} + (d_j - 2 - c_{ij})\cdot\frac{1}{d_i d_j d_k (d_j-1)} + c_{ij}\cdot(\frac{1}{d_i d_j d_k (d_i-1)} + \frac{1}{d_i d_j d_k (d_j-1)} + \frac{1}{d_i d_j d_k (d_i-1)(d_j-1)}) = \frac{1}{d_i d_j d_k}(\frac{d_i-2}{d_i-1} + \frac{d_j-2}{d_j-1} + \frac{c_{ij}}{(d_i-1)(d_j-1)})$. Note that c varies from 0 to $\min(d_i, d_j) - 2$, so the overall increment is less than $\frac{9}{4}\cdot\frac{1}{d_i d_j d_k}$.

► **Type IV**: $d_i > 2, d_j > 2, d_k > 2$. If the removed node is a neighbor of i but not connected to j or k, the increment is $\frac{1}{d_i d_j d_k}\cdot\frac{1}{d_i-1}$; if the removed node is a common neighbor of i and j but not connected to k, then the increment is $\frac{1}{d_i d_j d_k}\cdot(\frac{1}{d_i-1} + \frac{1}{d_j-1} + \frac{1}{(d_i-1)(d_j-1)})$; if the removed node is a common neighbor of i, j and k, the increment is $\frac{1}{(d_i-1)(d_j-1)(d_k-1)} - \frac{1}{d_i d_j d_k} = \frac{1}{d_i d_j d_k}\cdot(\frac{1}{d_i-1} + \frac{1}{d_j-1} + \frac{1}{d_k-1} + \frac{1}{(d_i-1)(d_j-1)} + \frac{1}{(d_i-1)(d_k-1)} + \frac{1}{(d_j-1)(d_k-1)} + \frac{1}{(d_i-1)(d_j-1)(d_k-1)})$. Similarly, we can get the overall increment for a triad of Type IV: $\frac{1}{d_i d_j d_k}\cdot(\frac{d_i-2}{d_i-1} + \frac{d_j-2}{d_j-1} + \frac{d_k-2}{d_k-1} + \frac{c_{ij}}{(d_i-1)(d_j-1)} + \frac{c_{ik}}{(d_i-1)(d_k-1)} + \frac{c_{jk}}{(d_j-1)(d_k-1)} + \frac{c_{ijk}}{(d_i-1)(d_j-1)(d_k-1)})$, where c_{ij} (or c_{ik}, c_{jk}) is the number of common neighbors of i and j (or i and k, j and k), not including i, j, k; and c_{ijk} is the number of common neighbors of i, j and k. The increment is less than $4\cdot\frac{1}{d_i d_j d_k}$.

Therefore, the total increment over all the triad is less than $\delta = \sum_{\text{Type II}}\frac{1}{d_i d_j d_k} + \frac{9}{4}\sum_{\text{Type III}}\frac{1}{d_i d_j d_k} + 4\sum_{\text{Type IV}}\frac{1}{d_i d_j d_k}$ and after normalizing it by $|\Delta_H|$ we get: $\mathbb{E}_H(\frac{1}{d_i d_j d_k}) < \mathbb{E}_G(\frac{1}{d_i d_j d_k}) + \frac{\delta}{(n-3)\Delta_G}$.

Next, we compute $m_{3,H}$:

$$
\begin{aligned}
m_{3,H} &= 2\,\mathbb{E}_H(\Delta_i)\,\mathbb{E}_H(\frac{1}{d_i d_j d_k}) \\
&< 2\frac{n-3}{n-1}\,\mathbb{E}_G(\Delta_i)(\mathbb{E}_G(\frac{1}{d_i d_j d_k}) + \frac{\delta}{(n-3)\Delta_G}) \\
&= 2(\frac{n-3}{n-1}\,\mathbb{E}_G(\Delta_i)\,\mathbb{E}_G(\frac{1}{d_i d_j d_k}) + \frac{\mathbb{E}_G(\Delta_i)\delta}{(n-1)\Delta_G}) \\
&= 2(\mathbb{E}_G(\Delta_i)\,\mathbb{E}_G(\frac{1}{d_i d_j d_k}) + \frac{\mathbb{E}_G(\Delta_i)\delta}{(n-1)\Delta_G} - \frac{2}{n-1}\,\mathbb{E}_G(\Delta_i)\,\mathbb{E}_G(\frac{1}{d_i d_j d_k})) \\
&= m_{3,G} + \frac{2\,\mathbb{E}_G(\Delta_i)}{(n-1)\Delta_G}(\delta - 2\Delta_G\,\mathbb{E}_G(\frac{1}{d_i d_j d_k})) \\
&= m_{3,G} + \frac{6}{n(n-1)}(\delta - 2\Delta_G\,\mathbb{E}_G(\frac{1}{d_i d_j d_k})) \quad (\text{as } E_G(\Delta_i) = \frac{3\Delta_G}{n}).
\end{aligned}
$$

Notice that $\Delta_G\,\mathbb{E}_G(\frac{1}{d_i d_j d_k}) = \sum_{(i,j,k)\in G}\frac{1}{d_i d_j d_k}$, so $\delta - 2\Delta_G\,\mathbb{E}_G(\frac{1}{d_i d_j d_k}) = \frac{1}{4}\sum_{\text{Type III}}\frac{1}{d_i d_j d_k} + 2\sum_{\text{Type IV}}\frac{1}{d_i d_j d_k} - \sum_{\text{Type II}}\frac{1}{d_i d_j d_k} - 2\sum_{\text{Type I}}\frac{1}{d_i d_j d_k}$. The theorem is proved. □

A.3 Proof of Theorem 4.6

Proof. Similarly, there are n subgraphs $G_1, G_2, \ldots, G_n$. We construct $H = \bigcup_{i=1}^{n} G_i$, so $m_{4,H} = \frac{\sum_i m_{4,G_i}}{n} = \mathbb{E}(m_{4,G'})$. From Theorem 4.2, $m_4 = [\,\mathbb{E}(d_i) + 4\,\mathbb{E}\binom{d_i}{2} + 2\,\mathbb{E}(\square_i)\,]\,\mathbb{E}(\frac{1}{d_i d_j d_k d_l})$. In the formula, $\mathbb{E}(d_i)$ is the average degree; $\mathbb{E}\binom{d_i}{2}$ is the average number of wedges a node is in, so it equals to $\frac{w}{n}$ where w is the number of wedges; $\mathbb{E}(\square_i)$ is the average number of squares a node is in. $\mathbb{E}(\frac{1}{d_i d_j d_k d_l})$ is the expected value of $\frac{1}{d_i d_j d_k d_l}$ over all the closed walks of length 4. Hence, for graph H, we have $m_{4,H} = [\,\mathbb{E}_H(d_i) + 4\,\mathbb{E}_H\binom{d_i}{2} + 2\,\mathbb{E}_H(\square_i)\,]\,\mathbb{E}_H(\frac{1}{d_i d_j d_k d_l})$.

From Equation 1, we have $\mathbb{E}_H(d_i) = \frac{n-2}{n-1}\cdot\mathbb{E}_G(d_i)$; As each wedge in G has $n-3$ copies in H (when none of the three nodes is removed), $\mathbb{E}_H\binom{d_i}{2} = \frac{w_H}{n(n-1)} = \frac{(n-3)\,w_G}{n(n-1)} = \frac{n-3}{n-1}\cdot\mathbb{E}_G\binom{d_i}{2}$; Similarly, $\mathbb{E}_H(\square_i) = \frac{n-4}{n-1}\cdot\mathbb{E}_G(\square_i)$ as each square in G has $n-4$ copies in H.

To get $\mathbb{E}_H(\frac{1}{d_i d_j d_k d_l})$, of course we can discuss all the possible closed walks of length 4 (generated by edges, wedges or squares), as we have done in the previous theorems. However, for a simpler exposition, we provide the following upper bound. Notice that $\mathbb{E}_H(\frac{1}{d_i d_j d_k d_l}) \le \mathbb{E}_G(\frac{1}{(d_i-1)(d_j-1)(d_k-1)(d_l-1)})$ and the bound is tight when G is a complete graph ($n \ge 3$) or all of its components are k-cliques ($k \ge 3$), and $\frac{1}{(d_i-1)(d_j-1)(d_k-1)(d_l-1)} = \frac{1}{d_i d_j d_k d_l}\cdot\prod_{x\in\{i,j,k,l\}}\frac{d_x}{d_x-1} \le \frac{16}{d_i d_j d_k d_l}$, so $\mathbb{E}_H(\frac{1}{d_i d_j d_k d_l}) \le 16\cdot\mathbb{E}_G(\frac{1}{d_i d_j d_k d_l})$. Therefore,

$$
\begin{aligned}
m_{4,H} &= [\,\mathbb{E}_H(d_i) + 4\,\mathbb{E}_H\binom{d_i}{2} + 2\,\mathbb{E}_H(\square_i)\,]\,\mathbb{E}_H(\frac{1}{d_i d_j d_k d_l}) \\
&\le [\,\frac{n-2}{n-1}\,\mathbb{E}_G(d_i) + 4\frac{n-3}{n-1}\,\mathbb{E}_G\binom{d_i}{2} + 2\frac{n-4}{n-1}\,\mathbb{E}_G(\square_i)\,] \\
&\quad \times 16\cdot\mathbb{E}_G(\frac{1}{d_i d_j d_k d_l}) \\
&\le \frac{n-2}{n-1}[\,\mathbb{E}_G(d_i) + 4\,\mathbb{E}_G\binom{d_i}{2} + 2\,\mathbb{E}_G(\square_i)\,] \times 16\cdot\mathbb{E}_G(\frac{1}{d_i d_j d_k d_l}) \\
&= \frac{16(n-2)}{n-1}m_{4,G}
\end{aligned}
$$

□